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ON DUAL L^1-SPACES AND INJECTIVE BIDUAL BANACH SPACES

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ABSTRACT

In a previous paper (Israel J. Math. **28** (1977), 313–324), it was shown that for a certain class of cardinals τ, $l^1(\tau)$ embeds in a Banach space X if and only if $L^1([0,1]^\tau)$ embeds in X^*. An extension (to a rather wider class of cardinals) of the basic lemma of that paper is here applied so as to yield an affirmative answer to a question posed by Rosenthal concerning dual $\mathscr{L}_1$-spaces. It is shown that if Z^* is a dual Banach space, isomorphic to a complemented subspace of an L^1-space, and κ is the density character of Z^*, then $l^1(\kappa)$ embeds in Z^*. A corollary of this result is that every injective bidual Banach space is isomorphic to $l^\infty(\kappa)$ for some κ. The second part of this article is devoted to an example, constructed using the continuum hypothesis, of a compact space S which carries a homogeneous measure of type ω_1, but which is such that $l^1(\omega_1)$ does not embed in $\mathscr{C}(S)$. This shows that the main theorem of the already mentioned paper is not valid in the case $\tau = \omega_1$. The dual space $\mathscr{C}(S)^*$ is isometric to

$$(L^1[0,1]^{\omega_1}) \oplus \Big(\sum_{\omega_1} {}^{\oplus} L^1[0,1] \oplus l^1(\omega_1)\Big)_1,$$

and is a member of a new isomorphism class of dual L^1-spaces.

1. Preliminaries

The notation and conventions used will be those of [3]. Cardinal numbers will be identified with the corresponding initial ordinals, but when the notation κ^λ is used it will be cardinal, rather than ordinal, exponentiation that is intended.

When μ is a measure, $\mathscr{L}^1(\mu)$ will denote the space of all μ-integrable functions, and $L^1(\mu)$ its quotient by the null functions. If f is in $\mathscr{L}^1(\mu)$, I write $f^\cdot$ for the corresponding element of $L^1(\mu)$. I write $\mathbf{D}$ for the two-point space $\{0,1\}$, and λ_A for the usual product measure on $\mathbf{D}^A$. If μ is any measure, the Banach space $L^1(\mu)$ is isometric to the l^1-direct sum

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$$\left(\sum_{a \in A} \oplus L^1(\nu_a)\right)_1,$$

where each ν_a is a finite measure. Moreover, it may be assumed that each $L^1(\nu_a)$ is isometric to $L^1(\lambda_{\kappa(a)})$ for a suitable cardinal $\kappa(a)$.

A Banach space X is said to be injective (or to be a $\mathcal{P}_\infty$-space) if, whenever Z is a Banach space, Y is a closed linear subspace of Z, and $T\colon Y \to X$ is a bounded linear operator, there exists a bounded linear operator $U\colon Z \to X$ which extends T. Most of the known results about injective Banach spaces are to be found in the paper [5] of Rosenthal; of the questions posed in that work, answers are given here to Conjecture 3 and 6 of §6.

The theory of injective Banach spaces is closely related to that of $\mathcal{L}_1$- and $\mathcal{L}_\infty$-spaces (by page 201 of [4]). Let us recall in particular that every injective Banach space is an $\mathcal{L}_\infty$-space, that Y is an $\mathcal{L}_1$-space if and only if Y^* is injective, and that Z is an $\mathcal{L}_\infty$-space if and only if Z^* is an $\mathcal{L}_1$-space. Thus, in studying injective bidual Banach spaces, we are simply looking at the second duals of $\mathcal{L}_\infty$-spaces. A vital tool is a lifting property possessed by dual $\mathcal{L}_1$-spaces, which appears as lemma 4 of [2], and which, for convenience, I record again here.

1.1 Lemma. *Let X and Y be Banach spaces and $J\colon X \to Y$ be a (linear homeomorphic) embedding. Let $Z \subseteq X^*$ be a closed linear subspace which is an $\mathcal{L}_1$-space. Then there is a closed linear subspace W of Y^* such that $J^* | W$ is a linear homeomorphism of W onto Z.*

As in [3] a crucial role in this paper will be played by a combinatorial lemma due to Erdös and Rado. Recall that a family of sets $(E(\alpha))_{\alpha \in \Delta}$ is said to be quasidisjoint (with common intersection I) if $E(\alpha) \cap E(\beta)$ is the same set I whenever α and β are distinct elements of Δ. It will be convenient for us to say that a cardinal τ has the property $(\ddagger)$ if $\kappa^\omega < \tau$ whenever κ is a cardinal and $\kappa < \tau$. The following result is theorem 1 of [1].

1.2 Lemma. *Let $(E(\alpha))_{\alpha \in \Gamma}$ be a family of countable sets, and suppose that $|\Gamma|$ is a regular cardinal with the property $(\ddagger)$. Then there is a subset Δ of Γ with $|\Delta| = |\Gamma|$ such that $(E(\alpha))_{\alpha \in \Delta}$ is quasidisjoint.*

2. Injective bidual Banach spaces

In this paragraph, a refinement of proposition 2.3 of [3] is presented, which allows us to prove a conjecture of Rosenthal's about injected biduals. First let us settle some notation. When B is a subset of A, we shall write π_B for the projection $\mathbf{D}^A \to \mathbf{D}^B$. In [3] use was made of the conditional expectation map $\mathcal{E}_B\colon L^1(\lambda_A) \to L^1(\lambda_B)$; it will be convenient here to work with maps

$$\mathcal{U}_B \quad \text{and} \quad \mathcal{V}_B : L^\infty(\lambda_A) \to L^\infty(\lambda_B)$$

defined by putting

$$\mathcal{U}_B f^\cdot = u^\cdot \quad \text{and} \quad \mathcal{V}_B f^\cdot = v^\cdot,$$

where u and v are given (almost everywhere) by

$$u(z) = \operatorname{ess\,sup}\{f(x,z) : x \in \mathbf{D}^{A\setminus B}\},$$

$$v(z) = \operatorname{ess\,inf}\{f(x,z) : x \in \mathbf{D}^{A\setminus B}\}.$$

We recall that every element of $L^1(\lambda_A)$ "depends on only countably many coordinates", in the sense that, if $f^\cdot \in L^1(\lambda_A)$ there exist a countable subset E of A and $g^\cdot \in L^1(\lambda_E)$ such that $f^\cdot = (g \circ \pi_E)^\cdot$. The following lemma expresses a by now familiar idea in what will be a convenient form.

2.1 Lemma. *Let B be a subset of A and $f^\cdot_\alpha = (g_\alpha \circ \pi_{E(\alpha)})^\cdot$ $(\alpha \in \Delta)$ be a bounded family of elements of $L^\infty(\lambda_A)$. Suppose that $E(\alpha) \cap E(\beta) \subseteq B$ whenever α and β are distinct elements of Δ. Write $u^\cdot_\alpha = \mathcal{U}_B f^\cdot_\alpha$, $v^\cdot_\alpha = \mathcal{V}_B f^\cdot_\alpha$, and suppose further that there exist real numbers r, and $\delta > 0$, such that the intersection*

$$F_M = \bigcap_{\alpha \in M} \{z \in \mathbf{D}^B : u_\alpha(z) > r + \delta, v_\alpha(z) < r\}$$

is non-null for every finite $M \subseteq \Delta$. Then the family $(f_\alpha)_{\alpha \in \Delta}$ is equivalent for the L^∞-norm to the usual basis of $l^1(\Delta)$.

Proof. It is enough, by proposition 4 of [6], to show that for every disjoint pair of finite subsets M_0, M_1 of Δ the set

$$G = \bigcap_{\alpha \in M_0} \{x \in \mathbf{D}^A : f_\alpha(x) > r + \delta\} \cap \bigcap_{\beta \in M_1} \{x \in \mathbf{D}^A : f_\beta(x) < r\}$$

is non-null. Using Fubini's theorem and the fact that the sets $E(\alpha)\setminus B$ are mutually disjoint, we can estimate $\lambda_A(G)$ by

$$\lambda_A(G) \geqq \int_{F_M} \left[\prod_{\alpha \in M_0} \lambda_{E(\alpha)\setminus B}\{y : g_\alpha(y,z) > r + \delta\} \right]$$

$$\left[\prod_{\beta \in M_1} \lambda_{E(\beta)\setminus B}\{y : g_\beta(y,z) < r\} \right] \lambda_M(dz).$$

Since the integrand is everywhere positive on the non-null set F_M (where $M = M_0 \cup M_1$), we see that $\lambda_A(G) > 0$.

2.2 PROPOSITION. *Let* $(\Omega, \mathcal{F}, \mu)$ *be a probability space and* τ *be a cardinal which satisfies* $(\ddagger)$ *and which is such that the cofinality* $\mathrm{cf}(\tau)$ *is either* ω *or else does itself satisfy* $(\ddagger)$. *Let* $(\dot{f}_\alpha)_{\alpha\in\Gamma}$ *be a family of elements of* $L^\infty(\Omega, \mathcal{F}, \mu)$ *satisfying* $|\Gamma| = \tau$, $\|\dot{f}_\alpha\|_{L^\infty} \leqq 1$ $(\alpha \in \Gamma)$, *and* $\|\dot{f}_\alpha - \dot{f}_\beta\|_{L^1} \geqq \varepsilon > 0$ $(\alpha, \beta \in \Gamma, \alpha \neq \beta)$. *Then there is a subset* Δ *of* Γ *with* $|\Delta| = \tau$ *such that* $(f_\alpha)_{\alpha\in\Delta}$ *is equivalent for the* L^∞*-norm to the usual basis of* $l^1(\Delta)$.

PROOF. For the case of τ a regular cardinal, this is proposition 2.3 of [3]. So we assume $\mathrm{cf}(\tau) = \kappa < \tau$ and find disjoint subsets of Γ, $\Gamma(\xi)$ $(\xi \in \kappa)$, in such a way that each $\tau(\xi) = |\Gamma(\xi)|$ is a regular cardinal, greater than κ and satisfying $(\ddagger)$, while we have also

$$\tau(\xi) > \sup\{\tau(\eta) : \eta < \xi\} \qquad \text{and}$$

$$\tau = \sup\{\tau(\xi) : \xi \in \kappa\}.$$

We may assume that the probability triple is in fact $(\Omega, \mathcal{F}, \mu) = (\mathbf{D}^A, \mathcal{B}(\mathbf{D}^A), \lambda_A)$, and that the functions f_α have the form $g_\alpha \circ \pi_{E(\alpha)}$ for suitable countable subsets $E(\alpha)$ of A and $\dot{g}_\alpha \in L^\infty(\lambda_{E(\alpha)})$. Using the combinatorial lemma of Erdös and Rado and various straightforward reductions of a type made in [3] (based on the observation that each $\tau(\xi)$ is a regular cardinal greater than 2^ω), we may assume that the sets $\Gamma(\xi)$ were chosen so that the following hold:

(i) each family $(E(\alpha))_{\alpha\in\Gamma(\xi)}$ is quasidisjoint, with common intersection $I(\xi)$, say;

(ii) for each ξ there are functions u_ξ, v_ξ in $\mathcal{L}^\infty(\lambda_{I(\xi)})$ such that

$$\mathcal{U}_{I(\xi)}\dot{f}_\alpha = \dot{u}_\xi \qquad \text{and}$$

$$\mathcal{V}_{I(\xi)}\dot{f}_\alpha = \dot{v}_\xi \qquad \text{for all } \alpha \in \Gamma(\xi).$$

Now we certainly have $\|\dot{u}_\xi - \dot{v}_\xi\|_{L^1} \geqq \varepsilon$ and $\|\dot{u}_\xi - \dot{v}_\xi\|_{L^\infty} \leqq 2$ so

$$\lambda_{I(\xi)}\{z : u_\xi(z) - v_\xi(z) \geqq \varepsilon/2\} \geqq \varepsilon/4.$$

If δ is any real with $0 < \delta < \varepsilon/2$ choose an integer $N > (\varepsilon/2 - \delta)^{-1}$. Then there exists an integer $M = M(\xi)$ with $|M(\xi)| \leqq N$, such that if we put $r = M/N$ we have

$$\lambda_{I(\xi)}\{z : u_\xi(z) > r + \delta, v_\xi(z) < r\} \geqq \varepsilon/8N.$$

We may suppose that the family of sets $(\Gamma(\xi))_{\xi\in\kappa}$ was so chosen that $M(\xi)$ is the same integer for all ξ.

We may now proceed inductively to make further refinements of the sets $\Gamma(\xi)$. For each ξ we know by hypothesis that

$$\Sigma(\xi)=\{E(\alpha)\colon \alpha\in\Gamma(\eta),\eta<\xi\}$$

is a set of cardinality strictly less than $\tau(\xi)$. Since $\tau(\xi)$ is a regular cardinal satisfying $(\ddagger)$, we may assume that the sets $\Gamma(\xi)$ were chosen so that the intersection $E(\alpha)\cap\Sigma(\xi)$ is the same countable subset of $\Sigma(\xi)$ for all $\alpha\in\Gamma(\xi)$. The last refinement is to note that (discarding at most κ elements from each $\Gamma(\xi)$) we may suppose that $E(\alpha)\cap E(\beta)\subseteq I(\xi)\cap I(\eta)$ whenever $\alpha\in\Gamma(\xi)$, $\beta\in\Gamma(\eta)$ and $\xi\neq\eta$. We are now ready to consider separately the two cases to be dealt with.

(i) $\mathrm{cf}(\tau)=\omega$

Put $B=\bigcup_{m\in\omega}I(m)$. Then certainly $E(\alpha)\cap E(\beta)\subseteq B$ whenever α,β are distinct elements of $\bigcup_{m\in\omega}\Gamma(m)$, and

$$\mathcal{U}_B f_\alpha^{\cdot}=(u_m\circ\pi_{I(m)})^{\cdot},$$

$$\mathcal{V}_B f_\alpha^{\cdot}=(v_m\circ\pi_{I(m)})^{\cdot}$$

whenever $\alpha\in\Gamma(m)$. We know that if

$$H_m=\{w\in\mathbf{D}^B\colon u_m(\pi_{I(m)}w)>r+\delta\quad\text{and}\quad v_m(\pi_{I(M)}w)<r\}$$

we have $\lambda_B(H_m)\geqq\varepsilon/8N$. Consequently there is an infinite subset σ of ω such that each intersection $\bigcap_{m\in M}H_m$, with M a finite subset of σ, is non-null. We put $\Delta=\bigcup_{m\in\sigma}\Gamma(m)$ and have the required result by Lemma 2.1.

(ii) $\mathrm{cf}(\tau)$ satisfies $(\ddagger)$

Since $\mathrm{cf}\,\tau)$ is regular we may assume that the family $(\Gamma(\xi))_{\xi\in\kappa}$ was chosen so that $(I(\xi))_{\xi\in\kappa}$ is quasidisjoint, with common intersection J, say. For each ξ we can choose a compact non-null subset K_ξ of $\mathbf{D}^{I(\xi)}$ such that $u_\xi(z)>r+\delta$, $v_\xi(z)<r$ for all $z\in K_\xi$. Making a last refinement, we may assume that $\pi_J[K_\xi]$ is the same compact subset of $\mathbf{D}^J$ for all ξ. If we put $B=\bigcup_{\xi\in K}I(\xi)$ and $\Delta=\bigcup_{\xi\in K}\Gamma(\xi)$, Lemma 2.1 is again applicable.

For our desired application of the above result we shall have need of some further ideas from [5] which for convenience are presented formally as lemmas.

2.3 Lemma. *Let X be a subspace of an L^1-space $(\Sigma_{a\in A}{}^{\oplus}L^1(\nu_a))_1$, where all the measures ν_a are finite. Let σ be the smallest cardinal of a subset B of A for which the natural map $P_B\colon X\to(\Sigma_{a\in B}{}^{\oplus}L^1(\nu_a))_1$ is a homeomorphic embedding. Then X has a complemented subspace isomorphic to $l^1(\sigma)$, but no subspace isomorphic to $l^1(\tau)$ for an uncountable cardinal $\tau>\sigma$.*

Proof. If C is a subset of A with $|C|<\sigma$ then there exists $x\in X$ with $\|x\|=1$ and $\|P_C x\|<\frac{1}{4}$. So we can find a finite subset D of A, disjoint from C, such that $\| P_D x \| > \frac{3}{4}$. In this way we can construct inductively a family $(x_\alpha)_{\alpha\in\sigma}$ of elements of ball X and a disjoint family $(D_\alpha)_{\alpha\in\sigma}$ of subsets of A such that $\| P_{D_\alpha} x_\alpha \| > \frac{3}{4}$ for all $\alpha \in \sigma$. That X has a complemented subspace isomorphic to $l^1(\sigma)$ now follows from lemma 1.1 of [5].

Now suppose that the family $(e_\alpha)_{\alpha\in\tau}$ of elements of ball X is equivalent to the usual basis of $l^1(\tau)$ for some uncountable $\tau > \sigma$. For suitably chosen finite subsets D_α of B the family $(P_{D_\alpha} e_\alpha)_{\alpha\in\tau}$ is still equivalent to the usual basis of $l^1(\tau)$. Now we can find an uncountable set Γ of indices α, such that D_α is the same finite set D, say, for all $\alpha \in \Gamma$. We deduce that the weakly compactly generated Banach space $\Sigma_{a\in D} L^1(\nu_a)$ has a subspace isomorphic to $l^1(\Gamma)$, which is false by remark 2 of §1 of [5].

2.4 Lemma. *If κ is a cardinal and X is a Banach space with a subspace isomorphic to $l^1(\kappa)$, then X^* has a subspace isomorphic to $l^1(2^\kappa)$.*

Proof. Let $I: l^1(\kappa) \to X$ be an embedding with transpose $I^*: X^* \to l^\infty(\kappa)$. By 1.1 it will be enough to prove that $l^1(2^\kappa)$ embeds in $l^\infty(\kappa)$. Since the compact space $\{0,1\}^{2^\kappa} = S$ has a dense subset of cardinality κ we can see that the space of continuous functions $\mathscr{C}(S)$ embeds in $l^\infty(\kappa)$. On the other hand, $l^1(2^\kappa)$ embeds in $\mathscr{C}(S)$ via the coordinate functions.

2.5 Theorem. *Let Z be an $\mathscr{L}_\infty$-space. If δ is the density character of Z^* then Z^* has a complemented subspace isomorphic to $l^1(\delta)$.*

Proof. Let S denote the unit ball of Z^* under the weak* topology, and I the natural embedding of Z in $\mathscr{C}(S)$. Then by 1.1 there is an embedding $J: Z^* \to \mathscr{C}(S)^*$ such that I^*J is the identity on Z^*. If $(\nu_a)_{a\in A}$ is a maximal family of mutually singular measures on S we can identify $\mathscr{C}(S)^*$ with

$$\left(\sum_{a\in A} \oplus L^1(\nu_a) \right)_1 .$$

Let B be a subset of A of minimal cardinality such that P_B is an isomorphism on Z^*. If $|B| = \delta$ Lemma 2.3 gives the desired result. Otherwise, let $\mu = |B|^+ \leqq \delta$. We shall obtain a contradiction to the second part of 2.3 by showing that Z^* has a subspace isomorphic to $l^1(\mu)$.

Firstly note that since μ is a regular cardinal it must be that one of the spaces $L^1(\nu_a)$ $(a \in B)$ has density character μ. Since ν_a is a measure on $S =$ ball Z^*, we have obvious operators

$$Z \to \mathscr{C}(S) \to L^\infty(\nu_a) \to L^1(\nu_a).$$

As in the proof of 2.6 of [3], we can conclude that the density character of the image of Z in $L^1(\nu_a)$ is μ, and hence that there exists a family $(z_\gamma)_{\gamma\in\mu}$ of elements of Z with

$$\|z_\gamma\|_Z = 1 \quad \text{and}$$

$$\|z_\beta - z_\gamma\|_{L^1(\nu_a)} \geqq \varepsilon > 0 \qquad (\beta \neq \gamma).$$

If the regular cardinal μ satisfies the condition ($\ddagger$) we are finished since, by 2.3 of [3], Z has a subspace isomorphic to $l^1(\mu)$. Hence by 2.4 Z^* has a subspace isomorphic to $l^1(2^\mu)$.

We now consider the case where μ does not satisfy ($\ddagger$). Let τ be the smallest cardinal such that $\tau < \mu$, $\tau^\omega \geqq \mu$. Then certainly τ satisfies ($\ddagger$) since if $\kappa^\omega \geqq \tau$ we have

$$\kappa^\omega = (\kappa^\omega)^\omega \geqq \tau^\omega \geqq \mu,$$

so that $\kappa \geqq \tau$, by choice of τ. It must also be the case that $\mathrm{cf}(\tau) = \omega$ since if $\mathrm{cf}(\tau) > \omega$ we have

$$\tau^\omega = \sup\{\kappa^\omega : \kappa < \tau\}.$$

Thus we can apply Proposition 2.2 and deduce that Z has a subspace isomorphic to $l^1(\tau)$. So Z^* has a subspace isomorphic to $l^1(2^\tau)$ and since $2^\tau \geqq \tau^\omega \geqq \mu$ the proof is finished.

2.6 Corollary. *Let X be an injective bidual Banach space. Then X is isomorphic to $l^\infty(\Gamma)$ for a suitable set Γ.*

Proof. As remarked in [5], this is an easy deduction from 2.5. If $X = Z^{**}$ is injective and $\delta = \mathrm{dens}\, Z^*$, then certainly X embeds as a subspace (necessarily complemented) of $l^\infty(\delta)$. On the other hand, we have just seen that Z^* has a complemented subspace isomorphic to $l^1(\delta)$, so that X has a complemented subspace isomorphic to $l^\infty(\delta)$. Now Pelczynski's decomposition method (or "accordion lemma"), proposition 1.4 of [5], gives the desired result.

3. An example

It was shown in [3] that, for a regular cardinal τ which satisfies ($\ddagger$), the following assertions about the Banach space X are equivalent:

(ai) X has a subspace isomorphic to $l^1(\tau)$;

(aii) X^* has a subspace isomorphic to $L^1(\lambda_\tau)$.

A closely related result was also given, that for a compact Hausdorff space T the following are equivalent (subject to the same restrictions on τ):

(bi) T carries a homogeneous measure of type τ;

(bii) there exists a continuous surjection from T onto $[0,1]^\tau$.

Subject to the generalized continuum hypothesis, a cardinal τ satisfies ($\ddagger$) if and only if it is not of the form $\tau = \kappa^+$ where $\mathrm{cf}(\kappa) = \omega$. The most obvious example, therefore, of a cardinal not satisfying ($\ddagger$) is ω_1, and I give in this paragraph an example to show that assuming the continuum hypothesis, neither of the above equivalences is valid for $\tau = \omega_1$. It also settles negatively conjecture 6 of [5] by showing that there is a dual L^1-space that is not isomorphic to an l^1-direct sum of spaces of the type

$$L_\kappa = \Big(\sum_{2^\kappa} {}^\oplus L^1(\lambda_\kappa)\Big)_1.$$

I have not been able to construct such an example without the use of CH.

3.1 Theorem. *Subject to the continuum hypothesis, there exist a compact space S and a measure μ on S such that the following hold:*

(i) $|S| = \omega_1$;

(ii) *μ is homogeneous of type ω_1;*

(iii) *every compact μ-null set is metrizable;*

(iv) *a nonzero measure ν on S is homogeneous of type ω_1 if and only if ν is absolutely continuous with respect to μ;*

(v) *$\mathscr{C}(S)^*$ is isometric to the l^1-direct sum*

$$(L^1([0,1]^{\omega_1})) \oplus \Big(\Big(\sum_{\omega_1} {}^\oplus L^1[0,1]\Big) \oplus l^1(\omega_1)\Big)_1;$$

(vi) *$\mathscr{C}(S)$ does not contain a subspace isomorphic to $l^1(\omega_1)$.*

Proof. The basic process in the construction is the following. Suppose that T is a compact space, that μ is a probability measure on T, and that $\mathscr{K} = (K_n)_{n\in\omega}$ is a sequence of disjoint closed subsets of T satisfying

$$K_n = \mathrm{supp}\,(\mu \mid K_n) \qquad (n \in \omega),$$

$$\mu\Big(\bigcup_{n\in\omega} K_n\Big) = 1.$$

Denote by $T^{\mathscr{K}}$ the subset

$$(T \times \{0\}) \cup \bigcup_{n\in\omega} (K_n \times \{2^{-n}\})$$

of $T \times \mathbf{R}$. Then $T^{\mathscr{K}}$ is compact and the map $p\colon T^{\mathscr{K}} \to T$; $(t,x) \mapsto t$ is continuous. We denote by $\mu^{\mathscr{K}}$ the measure on $T^{\mathscr{K}}$ obtained by splitting μ in half, that is

$$\mu^{\mathcal{K}} = \tfrac{1}{2} \sum_{n \in \omega} (\mu \mid K_n) \otimes (\delta(0) + \delta(2^{-n})).$$

Certainly the image $\tilde{p}(\mu^{\mathcal{K}})$ of $\mu^{\mathcal{K}}$ under p is μ and, if $T = \operatorname{supp} \mu$, then also $T^{\mathcal{K}} = \operatorname{supp} \mu^{\mathcal{K}}$.

The space S that we shall construct will be the inverse limit of a system

$$(S_\alpha, p_{\alpha\beta})_{\omega \leqq \alpha \leqq \beta < \omega_1}$$

of compact metrizable spaces indexed by the ordinals α with $\omega \leqq \alpha < \omega_1$. We shall also define probability measures μ_α on the spaces S_α; these will satisfy

$$\mu_\alpha = \tilde{p}_{\alpha\beta} \mu_\beta \qquad (\alpha < \beta)$$

and μ will be defined to be the inverse limit measure on S. As usual, we shall write p_α for the canonical map $S \to S_\alpha$.

We start by defining $S_\omega = \mathbf{D}^\omega$ and $\mu_\omega = \lambda_\omega$, and fix an enumeration $(N_\xi^\omega)_{\xi \in \omega_1}$ of the compact μ_ω-null subsets of S_ω. Suppose now that spaces S_β, continuous surjections $p_{\alpha\beta}$, measures μ_β, and enumerations $(N_\xi^\beta)_{\xi \in \omega_1}$ of the compact μ_β-null sets have been defined for all α, β with $\omega \leqq \alpha \leqq \beta < \delta$. In the case where δ is a limit ordinal we just take S_δ and μ_δ to be inverse limits, $p_{\alpha\delta}$ to be the naturally determined map and choose an enumeration $(N_\xi^\delta)_{\xi \in \omega_1}$ of the compact μ_δ-null sets.

If $\delta = \gamma + 1$, we note that the subset $E = \bigcup_{\alpha, \xi \leqq \gamma} p_{\alpha\gamma}^{-1}[N_\xi^\alpha]$ of S_γ is μ_γ-null, and choose a sequence $\mathcal{K} = \mathcal{K}(\gamma) = (K_n^\gamma)_{n \in \omega}$ of disjoint compact subsets of $S_\gamma \setminus E$ satisfying $K_n^\gamma = \operatorname{supp}(\mu_\gamma \mid K_n^\gamma)$,

$$\mu_\gamma\Big(\bigcup_{n \in \omega} K_n^\gamma\Big) = 1.$$

We take

$$S_{\gamma+1} = S_\gamma^{\mathcal{K}},$$

$$\mu_{\gamma+1} = \mu_\gamma^{\mathcal{K}},$$

$$p_{\gamma,\gamma+1} = p,$$

as in the basic process described above. We also define a map $r_\gamma: S_{\gamma+1} \to \{0, 1\}$ by

$$r_\gamma(s) = \begin{cases} 0 & \text{if } s \in S_\gamma \times \{0\}, \\ 1 & \text{otherwise.} \end{cases}$$

We may now turn our attention to proofs of the assertions (i) to (vi).

(i) We note that for each $z \in S_\omega$, $\{z\} = N_{\alpha(z)}^\omega$ for a suitable $\alpha(z) < \omega_1$. Hence, by the construction, $p_{\alpha(z),\beta}$ is injective on $p_{\omega,\beta}^{-1}(z)$ whenever $\beta \geqq \alpha(z)$, and so $p_{\alpha(z)}$ is injective on the subset $p_{\omega}^{-1}(z)$ of S. Since $S_{\alpha(z)}$ is a compact metrizable space, $|S_{\alpha(z)}| \leqq \omega_1$ and so $|p_{\omega}^{-1}(z)| \leqq \omega_1$. We deduce that $|S| = \omega_1$ from the equality

$$S = \bigcup_{z \in S_\omega} p_{\omega}^{-1}(z).$$

(ii) We can define a map $\rho: S \to \mathbf{D}^{\omega_1}$ by

$$(\rho s)_n = (p_\omega s)_n \qquad (n < \omega),$$

$$(\rho s)_\gamma = r_\gamma(p_{\gamma+1} s) \qquad (\omega \leqq \gamma < \omega_1).$$

Then ρ is Baire measurable and induces an isometry of $L^1(\mu)$ onto $L^1(\lambda_{\omega_1})$. Hence μ is homogeneous of type ω_1.

(iii) Let F be a compact μ-null subset of S. Then there exists $\alpha < \omega_1$ such that $p_\alpha[F]$ is μ_α-null. Hence $p_\alpha[F] = N_\xi^\alpha$ for some $\xi < \omega_1$. If $\gamma = \max\{\alpha, \xi\}$ then $p_\gamma \mid F$ is injective and F is therefore metrizable.

(iv) If ν is a nonzero measure on S which is singular with respect to μ, there is a compact subset F of S with $\mu(F) = 0$, $\nu(F) \neq 0$. By (iii), F is metrizable. Since a compact metrizable space cannot carry a measure of type ω_1, ν is not homogeneous of type ω_1.

(v) Let $(\nu_a)_{a \in A}$ be a maximal family of nonzero atomless measures on S, which are mutually singular, and singular with respect to μ. Then $\mathscr{C}(S)^*$ is isometric to

$$\left(L^1(\mu) \oplus \sum_{a \in A} {}^{\oplus} L^1(\nu_a) \oplus l^1(\omega_1)\right)_1.$$

It will be enough to prove that $|A| = \omega_1$, since $L^1(\mu)$ is isometric to $L^1(\lambda_{\omega_1})$ (or, equivalently, to $L^1([0,1]^{\omega_1})$), and each ν_a is of type ω (so that $L^1(\nu_a)$ is isometric to $L^1[0,1]$). I shall show, in fact, that there are only ω_1 measures of type ω on S. If ν is any such measure, ν is carried by some μ-null $\mathscr{K}_\sigma$ subset F of S. There is an ordinal $\alpha = \alpha(\nu) < \omega_1$ with the property that p_α is injective on $p_\alpha^{-1} p_\alpha[F] = F$. Consequently, if ν' is a measure on S and $\tilde{p}_\alpha \nu' = \tilde{p}_\alpha \nu$ we have $\nu' = \nu$. That is to say, the map $\nu \to (\alpha(\nu), p_{\alpha(\nu)}(\nu))$, which takes the set of measures of type ω on S into

$$\omega_1 \times \left(\bigcup_{\alpha < \omega_1} \mathscr{C}(S_\alpha)^*\right)$$

is injective.

(vi) By (v) and Lemma 2.3, $\mathscr{C}(S)^*$ does not have a subspace isomorphic to $l^1(2^{\omega_1})$. Hence by Lemma 1.1, $\mathscr{C}(S)$ does not have a subspace isomorphic to $l^1(\omega_1)$.

3.2 REMARK. The construction given in 3.1 shows clearly the way in which Proposition 2.2 (or proposition 2.3 of [3]) fails in the case $\tau = \omega_1$. We choose continuous functions f_α on S which are close approximations in $L^1(\mu)$ norm to the functions r_α. There is no uncountable set of indices for which the family $(f_\alpha)_{\alpha \in \Delta}$ is equivalent to the usual basis of $l^1(\Delta)$.

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